NOTE FOR MATH 126

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1. Theory of distribution

In order to apply the notion of derivatives to get general solution for a PDE, we introduce the notion of distributions.Firstly, we study an example of transport equation to understand the importance of doing this.

Example 1.1. *Given transport equation as*

$$\begin{cases} \partial_t u + v\nabla_\nu u = 0 \\ u(0,x) = u_0(x) \end{cases}$$

This can also be wrote as $(1,v)\nabla_{t,x}u = 0$, which means $u(t,x) = u_0(x-tv)$ is a solution. However, this induction process requires $u \in C^1 \Leftrightarrow u_0 \in C^1$, and is unique given $u_0 \in C^1$. But what can we do if we have $u \in C$?

1.1. Distributions.

We can introduce smooth functions to manipulate the equation in other to get a broader sense of function and taking derivatives.

$$0 = \int \phi\left[\partial_t u + v\nabla_\nu u\right] dxdt = -\int u\left[\partial_t \phi + v\nabla_\nu \phi\right] dxdt,$$

which still makes sense for $u \in C$. And the equal sign holds iff u satisfies the equation.

Def 1.2. ***Test Functions*** $\mathcal{D} \triangleq \{\phi : \mathbb{R}^n \to \mathbb{R}, u \in C^\infty$ *with compact support*$\}$.

Therefore, we begin to give definition of distributions.

Def 1.3. *A* ***distribution*** *f is a continous, linear map $f : \mathcal{D} \to \mathbb{R}$.*

And we define ***continuous*** *by defining* ***convergence*** *in $\mathcal{D}$, which means*

- *$\phi_n \to \phi$ uniformly.*
- *$\partial\phi_n \to \partial\phi$ uniformly.*
- *There exist K compact, s.t $Supp\phi_n, Supp\phi \subseteq K$.*

Distribution is unique when viewed as a continuous functions for we can always use smooth function which supported near the x thet $u(x) \neq v(x)$.

Remark: Not all distributions are continuous, as we can see the Heaviside function (or any step function that jumps).

1.2. Derivatives of ditributions.

Now, with the theory of distributions we can take derivatives of continuous functions by viewing them as distributions. Generally, given $u \in C^1$, we have $u'(\phi) = \int u'\phi dx = -\int u\phi' dx$.

Def 1.4. *Now we give the definition for* ***derivatives of distributions****: $u'(\phi) \triangleq -u(\phi')$. This makes sense for $u \in \mathcal{D}$.*

Now we emphasize on the importace of Dirac Mass.

Example 1.5. *Given* ***Heaviside function***

$$H(x) = \begin{cases} 1 & x > 0 \\ 0 & x \leqslant 0 \end{cases}$$

Obviously $H \in \mathcal{D}'$ and we are about to show that

$$H'(\phi) = -H(\phi') = -\int_0^{+\infty} \phi'(x)dx = \phi(0) = \delta_0(\phi).$$

This implies $H' = \delta_0$. Also, We have $\delta_0'(\phi) = -\delta_0(\phi') = -\phi'(0)$. This means H is infinitely differentiable in a distribution sense.

1.3. **Multiply distribution by smooth function.** We take $f \in \mathcal{D}'$, $\alpha \in C^\infty$, what does $f * \alpha$ means?

- If f is a function, $\alpha * f =$ multiplication, and
$$\alpha f(\phi) = \int \alpha(x) f(x) \phi(x) dx = \int f(x)(\alpha\phi)(x) dx,$$
which means
$$(af)(\phi) = f(a\phi).$$
- If we have $P(x, \partial) = \sum_{|\alpha| \leqslant m} c_\alpha(x) \partial^\alpha$, $C_\alpha(x) \in C^\infty$, $u \in \mathcal{D}'$, so
$$P(x, \partial) u = f$$
is meaningful.

Def 1.6. *Given u, v, a **convolution** is*
$$u * v(x) = \int_{-\infty}^{+\infty} u(y) v(x-y) dy.$$

And the definition makes sence if $u, v \in \mathcal{D}$, so $u * v(x) \in \mathcal{D}$ and it can be justified by property after.

Property 1.7. *For convolution:*

- $u * v = v * u$.
- $\partial_j(u * v) = u * \partial_j v = \partial_j u * v$.
- $(u * v) * w = u * (v * w)$.

This means after convolution, it is still compact(compact + compact) and infinitely differentiable.

Extend the concept:

(1) Consideridng convolution on $\mathcal{D} * \mathcal{D}' : u * v(x) = \int_{-\infty}^{+\infty} u(y) v(x-y) dy := v(u(x - \bullet))$.
Define the **support of a distribution**: $v \in \mathcal{D}'$, $x_0 \notin supp v$if $v(\phi) = 0$ whenever ϕ is supported in a small neighbourhood of x_0. For example, $supp\delta_0 = \{0\}$.

(2) For a random distribution, we don't know what the convolution will behave in ∞. Considering the **compactly supported distribution** $\mathcal{E}'$, we have $\mathcal{D} * \mathcal{E}' \to \mathcal{D}$.

(3) $\mathcal{E}' * \mathcal{E}' \to \mathcal{E}'$:
$$\begin{aligned}(u * v)(\phi) &= \int \int u(y) v(x-y) dy \phi(x) dx \\ &= \int \int u(y) v(z) \phi(y+z) dy dz \\ (u * v)(\phi) &= u_y(v(\phi(y + \bullet))).\end{aligned}$$
So it extends to distribution.

Algebra:

- $(\mathcal{D}, +, \cdot)$ is an algebra, may extends to $\mathcal{E}$, which means it now has an identity $u = 1$.
- $(\mathcal{D}, +, *)$ is an algebra, may extends to $\mathcal{E}'$ ($\mathcal{D}$ is dense in $\mathcal{E}'$ when viewed as distributions), has an identity $f = \delta_0$: $(\delta_0 * \psi)(x) = \delta_0(\psi(x - \bullet)) = \psi(x)$.

1.4. **Foundamental Theorem in Calculus in a distributional sense.** Here we want to know what distributions of derivatives 0 looks like.

Given $u' = f$, $f \in \mathcal{D}'$. Therefore, we have $u(\phi') = -u'(\phi) = -f(\phi)$. Therefore, if ϕ is the derivative of a test function ψ then $u' = 0$ means $u'(\phi) = -0(\psi) = 0$, which means $u' = 0$ in a distributional sense.

Question: when can we get a test function ϕ by take derivative of a smooth Ψ? In oder the integral of ϕ to be a compactly supported smooth function, this

$$\Leftrightarrow \int_{-\infty}^{+\infty} \phi(x)dx = 0.$$

Let $\phi_0 \in \mathcal{D}$ with $\int \phi_0 = 1$, then we can write

$$\phi_1 = \phi - c\phi_0, c = \int \phi(x)dx.$$

Threfore,

$$u(\phi) = u(\phi_1) + cu(\phi_0).$$

And we attribute a value for $u(\phi_0)$.

Proposition 1.8. *A distribution u satisfies $u' = 0$, then $u =$ constant.*

Proof If we have

$$\begin{cases} u_1' = f \\ u_2' = f \end{cases},$$

for $u = u_1 - u_2$, $u' = 0$, therefore $u(\phi) = 0$ for any ϕ satisfy $\int_{-\infty}^{+\infty} \phi(x)dx = 0$. For arbitrary ϕ, $u(\phi) = u(\phi_0)\int \phi dx = u(\phi_0)1(\phi)$, which means $u = u(\phi_0)$. $\square$

1.5. **Using Foundamental Solution to Conpute Distributions.**

Theorem 1.9 (What are really distributions?)**.** *For any $f \in \mathcal{D}'$ and any compact K we have a representation*

$$f = \sum_{|\alpha| \leqslant m} \partial^\alpha u_\alpha(x), x \in K, u_\alpha \in C.$$

We would notify $f \in C^{-m}$.

Given $u' = f$, $f \approx \sum_n f(x_j) 1_{[x_j, x_{j+1}]} = \sum_n \delta_{x_j} f(x_j)(x_{j+1} - x_j) \overset{n\to\infty}{=} \int f(y)\delta_y(x)dy$. Then we can write $u(x) = \int f(y)H(x-y)dy$.

Def 1.10. *Since $H' = \delta_0$, H is the* ***foundamental solution*** *for ∂ in $\mathbb{R}$. If we have $\partial u = f$, we get*

$$u(x) = \int f(y)H(x-y)dy.$$

However, the solutions and foundamental solutions cannot be unique due to constants. Generally, given $P(\partial) = \sum_{|\alpha| \leqslant m} c_\alpha \partial^\alpha$, c_α constants, we define foundamental solution as

Def 1.11. *$K \in \mathcal{D}'$ is a* ***foundamental solution*** *if*

$$P(\partial)K = \delta_0.$$

Proposition 1.12. *If K is a foundamental solution for $P(\alpha)$, then a solution to $P(\alpha)u = f$ is given by $u = K * f$.*

Proof $P(\partial)(K * f) = \sum c_\alpha \partial^\alpha (K * f) = P(\partial)K * f = \delta_0 * f = f$. $\square$

Q1) Do foundamental solutions exist?

1) For counstant coefficient PDEs they do exist.

2)But there are variable coefficient PDEs which are not solvable.

Q2) Are foundamental solutions unique?

Consider $P(\partial)u = 0$, we look for $u = e^{\sum \xi_j x_j}$. So, $P(\partial)u = P(\xi)u$, it leaves to solve $P(\xi) = 0$. We need $\xi \in \mathbb{C}$ then it must have roots, and we have **bounded** solutions for homogeneous problem.

2. Solving Laplace Equation

2.1. **The Laplace Equation.** Given by

$$-\Delta u = f.$$

Example 2.1 (A small insight)**.** *Considering in 1-dimension,*

$$-\Delta u = -\partial^2 u = \delta_0,$$

then we get $u(x) = \begin{cases} 0 & x \text{ ¡ } 0 \\ -x & x \text{ ¿ } 0 \end{cases}$ *(forward foundamental solution, supported in* $[0, +\infty)$*), which is not unique for we can add* $ax + b$*. Backward foundamental solution* $K^{-}(x) = \begin{cases} x & x \text{ ¡ } 0 \\ 0 & x \text{ ¿ } 0 \end{cases}$*. In general, we get* ***symmetric foundamental solution*** $K^{sgn}(x) = \frac{1}{2}(K^{+}(x) + K^{-}(x)) = -\frac{1}{2}|x|$.

Theorem 2.2. $-\partial^2 u = f$ *has the solutions* $u = f * K + ax + b$*. If* $f \in \mathcal{E}'$*,* $suppf * K^{+} \subset$ *compact set* $+[0, +\infty)$*, so* $u = 0$ *means* $-\infty$*,* $suppf * K^{+} \subset$ *compact set* $+(-\infty, 0]$*, so* $u = 0$ *means* $+\infty$*.*

Change of coordinates [From now on, we will use Einstein Convention to omit summation]: If $y = Ax$ given by $y_j = A_{j,k}x_k$, therefore we have

$$\frac{\partial}{\partial x_j} = \frac{\partial y_k}{\partial x_j}\frac{\partial}{\partial y_k} = A_{k,j}\frac{\partial}{\partial y_k},$$

which means

$$\Delta = \frac{\partial}{\partial x_j}\frac{\partial}{\partial x_j} = A_{l,j}A_{k,j}\frac{\partial}{\partial y_l}\frac{\partial}{\partial y_k},$$

in which $A_{l,k}A_{k,j} = A_{l,k}A'_{j,k}$ if A is orthogonal (and this is true for change of coordinates which is rotation). This suggests Δ is invariable under rotation, and we are about find foundamental solution under rotations. We are going to use spherical coordinates: $K = K(r)$, $r = |x|$.

$$\begin{aligned} -\Delta K(r) &= -\partial_j \partial_j K(r) \\ &= -K''(r) - \frac{n-1}{r}K'(r). \end{aligned}$$

To solve $-\Delta K = \delta_0$ means $-\Delta K = 0$ when away from $x = 0$, which means

$$K = \begin{cases} Cr^{2-n} + D & n \geqslant 3 \\ Clnr + D & n \leqslant 2 \end{cases}$$

Proposition 2.3 (Green's theorem)**.** *Given an area* D*,*

$$\int_D \Delta uv dx = -\int_D \partial_j u \partial_j v dx + \int_{\partial D} \nu_j \partial_j uv dA \tag{1}$$

$$= \int_D u\partial_j\partial_j v dx + \int_{\partial D} \nu_j(\partial_j uv - \partial vu) dA \tag{2}$$

$$= \int_D u\Delta v dx + \int_{\partial D} \frac{\partial u}{\partial \nu}v - \frac{\partial v}{\partial \nu}u dA. \tag{3}$$

This implies Δ has some kind of "symmetric" operator, and $\frac{\partial u}{\partial \nu}$ is called normal derivative.

Now we find C s.t $-\Delta K = \delta_0$.Compute $\Delta K(r)$ in $\mathcal{D}'$:

$$\begin{aligned}
\Delta K(\phi) &= K(\Delta\phi) \\
&= \int_{\mathbb{R}^n} \Delta\phi K dx \\
&= \int_{\mathbb{R}^n - B_\epsilon} \Delta\phi K dx + \int_{B_\epsilon} \Delta\phi K dx \\
&= \int_{\mathbb{R}^n - B_\epsilon} \phi\Delta K(=0) dx + \int_{\partial(\mathbb{R}^n - B_\epsilon)} \frac{\partial\phi}{\partial\nu} K dA[I] - \int_{\partial(\mathbb{R}^n - B_\epsilon)} \frac{\partial K}{\partial\nu}\phi[II] + \int_{B_\epsilon} \Delta\phi K dx[III]
\end{aligned}$$

As $\epsilon \to 0$, [I] $\leqslant C\epsilon^{n-1}\epsilon^{2-n}$ $o(\epsilon)$ or $C\epsilon ln(\epsilon)$, [III] is bounded on a volume of $C\epsilon^n$, therefore goes to 0. For [II], $\frac{\partial K}{\partial \nu} = -K'(r) = \begin{cases} -(2-n)r^{1-n} & n \geqslant 3 \\ -\frac{1}{n} & n = 2 \end{cases}$ $\phi(x) = \phi(0) + o(\epsilon)$ so

$$II = (\phi(0) + o(\epsilon))[(2-n)\epsilon^{1-n}]C_n\epsilon^{n-1} \to (2-n)\phi(0).$$

This means

$$K(x) = \begin{cases} \frac{1}{(2-n)C_n} r^{2-n} & = \frac{1}{n(n-2)\alpha(n)} |x|^{2-n} & n \geqslant 3 \\ \frac{1}{C_2} ln r & = -\frac{1}{2\pi} ln(|x|) & n = 2 \end{cases}$$

Example 2.4. *To tackle the problem of uniqueness of K, we consider **harmonic function** to address $\Delta(K_2 - K_1) = 0$. Recall that z^n are always harmonic so $Re z$ and $Im z$ are always harmonic.Examples in 2-dimension: $x, y, x^2 - y^2, xy, x^3 - 3xy^2$ and so on.*

2.2. **Mean Value Property.** Given a ball $B_r(x_0)$, $S_r = \partial B_r(x_0)$, we define mean value symbol as $⨍_D u = \frac{1}{|D|}\int u$, and $u(x_0) = ⨍_{B_r} u(x)dx = ⨍_{S_r} u(x)dA$ holds for harmonic functions (it holds for boundary and take unions to hold for balls):

Considering Green's Theorem:

$$\int_D u\Delta v dx = \int_D v\Delta u dx + \int_{\partial D} \frac{\partial v}{\partial \nu}u - \frac{\partial u}{\partial \nu} v dA$$

If we put $\Delta v = \delta_{x_0}$, letting $v = 0$ on ∂B_r to get $v = -K(x - x_0) + K(r)$. Therefore, since v is radical, $\frac{\partial v}{\partial \nu}$ is constant and equals to $\frac{1}{\partial D}$ by taking $u = 1$:

$$u(x_0) = \int_{\partial D} u \frac{\partial v}{\partial \nu} dA = \frac{1}{\partial D}\int_{\partial D} u dA.$$

For general $-\Delta u = f$ we get

$$u(x_0) = -\int_D v f dx + ⨍_{\partial B(x_0,r)} u(x) dA.$$

Suppose $f \geqslant 0$, then $u(x_0) \geqslant ⨍_{\partial B(x_0,r)} u(x)dA$; if $f \leqslant 0$, then $u(x_0) \leqslant ⨍_{\partial B(x_0,r)} u(x)dA$.

Def 2.5.

$$\begin{array}{lll} -\Delta u = f \geqslant 0 & u & \textit{is superharmonic} \\ -\Delta u = f \leqslant 0 & u & \textit{is subharmonic} \end{array}$$

Theorem 2.6. *Given $u \in C^2$ has mean value property, then u is harmonic.*

Proof Suppose $-\Delta u(x_0) > 0$ then it still holds in a small $B(x_0, r)$, so $u(x_0) > ⨍_{\partial B(x_0,r)} u(x)dA$ and contradicts. $\square$

2.3. String maximum principal.

Theorem 2.7 (Weak maximum principal). *Given $-\Delta u = 0$ in D connected. Suppose $u \in C^2(D)$ and $u \in C(\overline{D})$, then*

$$\max_{x \in \overline{D}} u(x) = \max_{x \in \partial D} u.$$

Further more , (Strong maximum principal) if max is attained inside, then $u \equiv$ constant (This requires connected)

Proof If the max can be attained at some point $x_0 \in D$, then take $B(x_0, r) \subset D$, then

$$\max u = u(x_0) = \fint_{B(x_0,r)} u(x)dx \leqslant \fint_{B(x_0,r)} u(x_0)dx = u(x_0).$$

This means $u(x) = u(x_0)$ in $B(x_0, r)$.Then by taking path (connected equals path-connected in $\mathbb{R}^n$)from x_0 to a random x_1 and cover it with balls (or we can prove this by proving that $E = \{x : u(x) = u(x_0)\}$ is closed and open in D). Note: this still holds for subharmonic functions, and the Minimum Principal will hold for superharmonic functions. □

2.4. In larger space.

Condsidering $-\Delta u = f$ in $\mathbb{R}^n$, and we consider the equation firstly in a domain D.

Example 2.8 (Dirichlet boundary value problem).

$$\begin{cases} -\Delta u &= f \quad x \in D \\ \quad u &= g \quad x \in \partial D \end{cases}$$

Neumann boundary condition is $\frac{\partial u}{\partial \nu} = g$ on ∂D.

Proposition 2.9. *The Dirichlet problem has at most one solution.*

Proof Suppose we have two solutions u_1, u_2, so $u_1 - u_2 = 0$ on boundary and $\Delta(u_1 - u_2) = 0$ by maximum and minimum principals we have $u_1 = u_2$. □

[Perron's method squizing]

2.5. Elliptic regularity.

Theorem 2.10. *All harmonic functions are smooth (which means C^∞).*

Proof We can give several proofs.

Proof 1 Elliptic regularity means K is smooth away from 0.

Claim 1 Suppose $u \in C^2$ has compact support and $-\Delta u = f$, $u = K * f$.
Since $-\Delta(u - K * f) = -\Delta v = 0$, u has compact support and $\widetilde{u} = K * f$ may not have compact support but tends to 0 as $x \to \infty$. This means v tends to 0 as $x \to \infty$ and by applying maximum and minimum principals we get $v = 0$.

Claim 2 Given harmonic function $u \in C^2(D)$, we need to prove it's smooth in D.
Choose x_0 s.t $B(x_0, 2r) \subset D$, we show that u is smooth in the ball. Taking a cut-off function χ which is 1 in $B(x_0, r)$ and smoothly go down to 0 utill it is 0 outside $B(x_0, 2r)$. We take $v = \chi u$,

$$-\Delta v = -u\Delta\chi - 2\nabla\chi\nabla u = f.$$

Since *Suppf* $\subset B(x_0, 2r)/B(x_0, r)$, we have $v(x) = \int f(y)K(x - y)dy$. $x \in B(x_0, r)$ and $y \in B(x_0, 2r)/B(x_0, r)$ we have $|x - y| > c$, and K is smooth away from 0 so the integral is well defined. We can differentiate as many times as we want inside the integral.

Proof 2 Dirtributions's View by using Weierstrass theorem: Any continuous functions on a interval can be uniformly approximated by smooth functions + polynimial functions + trigonometric functions.

Given $u \in C$ $(dim = n)$, $u * \delta_0 = u$ in distribution theory. Now we approximate δ_0 by smooth functions. Take $\psi \in \mathcal{D}$, $\int \psi = 1$, take $\psi_\epsilon = \frac{1}{\epsilon^n}\psi(\frac{x}{\epsilon})$ we still have $\int \psi_\epsilon = 1$.
Claim: $\psi_\epsilon \to \delta_0$ in $\mathcal{D}'$:

$$\psi_\epsilon(\phi) = \int \frac{1}{\epsilon^n}\psi(\frac{x}{\epsilon})\phi(x)dx = \int \psi(y)\phi(\epsilon y)dy \to \int \psi(y)\phi(0) = \phi(0).$$

This means $u_\epsilon = u * \psi_\epsilon \to u * \delta_0 = u$ in $\mathcal{D}'$, and $u_\epsilon \in C^\infty$.

Suppose u is harmonic, $u_\epsilon(x) = \int u(y)\psi_\epsilon(x-y)dy = u(x)$ (we can choose ψ to be radially symmetric) by mean value theorem [**Evans P28**].

□

Question: Given u in $B(0,r)$, what can we say about $u'(0)$?

$$\partial_1 u(0) = ⨍_{B(0,r)} \partial_1 u(x)dx = \frac{C_n}{r^n}\int_{\partial B(0,r)} \nu_1 u, dx$$

$$|\partial_1 u(0)| \leqslant \frac{C_n}{r^n}\sup_{\partial B(0,r)}|u|\, r^{n-1} = \frac{C_n}{r}\sup_{\partial B(0,r)}|u|\,.$$

Theorem 2.11 (Liouville)**.** *The only bounded harmonic functions are the constants.*

Proof Suppose $|u| \leqslant M$,

$$|\partial_j u(y)| \leqslant \frac{C_n}{r}\sup_{\partial B(y,r)}|u| \leqslant \frac{C_n}{r}M.$$

As $r \to \infty$, RHS tends to 0 so u can only be constant. □

Proposition 2.12.

$$\left|\partial^k u(x_0)\right| \leqslant \frac{C_{n,k}}{r^k}\sup_{x\in B(X_0,r)}|u(x)|\,.$$

Proof Given $\psi_r(x) = \frac{1}{r^n}\psi(\frac{x}{r})$, $\psi \in \mathcal{D}$, $\int \psi = 1$, $supp\psi_r$ in $B(0,r)$. If $u \to u_r = u * \psi_r$, u is harmonic so $u = u_r$. As we can take derivative on both sides of convolution, we have $\partial^k u(x_0) = u * \partial^k \psi_r(x)$. This means

$$\left|\partial^k u(x_0)\right| \leqslant \int_{B(0,r)} |u(x_0-y)|\left|\partial^k\psi_r(y)\right| dy \leqslant \sup_{B(x_0,r)}|u|\int_{B(0,r)}\left|\partial^k\psi_r(y)\right| dy.$$

since

$$\partial^k\psi_r(x) = \frac{1}{r^{n+k}}(\partial^k\psi)(\frac{x}{r}),$$

the integral is (we can choose good ψ)

$$\int \frac{1}{r^{n+k}}\left|(\partial^k\psi)(\frac{x}{r})\right| dx = \frac{1}{r^k}\int \left|(\partial^k\psi)(y)\right| dy = \frac{1}{r^k}C_{n,k}.$$

□

Theorem 2.13 (Liouville)**.** *Suppose u is harmonic in $\mathbb{R}^n$, and $|u(x)| \leqslant C(1+|x|^N)$, therefore u is a polynomial of degree $\leqslant N$.*

Proof

$$\left|\partial^{N+1}u(x_0)\right| \leqslant \frac{1}{r^{N+1}} \sup_{B(X_0,r)} |u| \leqslant \frac{C}{r^{N+1}}(1+(|x_0|+r)^N) \to 0.$$

□

The Midterm 1

2.6. **Green's Function for the Dirichlet Problem.** Given $\Omega \in \mathbb{R}^k$ is a bounded nice area,

$$\begin{cases} -\Delta u &= f & \text{in} & \Omega \\ u &= g & \text{in} & \partial\Omega \end{cases}$$

Try to use the foundamental solution, we have

$$\begin{aligned}\int_\Omega -\Delta u K(x-y)dy &= \int_\Omega f(y)K(x-y)dy \\ &= \int_\Omega u(-\Delta K(x-y))dy - \int_{\partial\Omega} \frac{\partial u(y)}{\partial \nu}K(x-y) - \frac{\partial K(x-y)}{\partial \nu}g(y)d(\nabla y).\end{aligned}$$

Therefore we get

$$\int_{\mathbb{R}^k} u\delta_x dy = u(x) = \int_\Omega f(y)K(x-y)dy - \int_{\partial\Omega} \frac{\partial u(y)}{\partial \nu}K(x-y) - \frac{\partial K(x-y)}{\partial \nu}g(y)d(\nabla y).$$

If no boundary and $u \to 0$ at ∞, then $u(x) = \int f(y)K(x-y)$. If we have a boundary, we don't know $\frac{\partial u}{\partial \nu}$ on $\partial\Omega$. Therefore, we need to give stronger condition of the fundamental solution to get $G(x,y)$, and we still want $-\Delta G(x,y) = \delta_x$. If $G(x,y) = 0$ on $\partial\Omega$, then

$$u(x) = \int_\Omega f(y)G(x-y)dy + \int_{\partial\Omega} \frac{\partial G(x,y)}{\partial \nu}g(y)d(\nabla y). \tag{4}$$

Therefore, we need to find the **Green Function** for Ω which satisfies

1 $-\Delta_y G(x,y) = \delta_x$.
2 $G(x,y) = 0$, $y \in \partial\Omega$.

We look for $G(x,y) = K(x-y) + \phi(x,y)$ and satisfies

1 $-\Delta_y \phi(x,y) = 0$.
2 $\phi(x,y) = -K(x-y)$, $y \in \partial\Omega$.

This means ϕ is harmonic and smooth on the boundary, it's natural to expect ϕ is smooth.We choose $u(y) = G(z,y)$ in dsj to get

$$\begin{aligned}u(x) &= \int_\Omega G(x,y)(-\Delta u(y))dy + \int_{\partial\Omega} \frac{\partial G(x,y)}{\partial \nu}u(y)d(\nabla y) \\ &= u(x) = \int_\Omega G(x,y)\Delta_z dy + \int_{\partial\Omega} \frac{\partial G(x,y)}{\partial \nu} \times 0d(\nabla y) = G(z,y).\end{aligned}$$

This means Green function is symmetric and so is ϕ. And we claim that

1 A Green function exists in any C^1 domain.
2 Green function can be explicitly computed in some single domains.

Example 2.14 (Half-Plane). *Given $H = \{x_n > 0\} \subset \mathbb{R}^n$, $x = (x_1, \cdots, x_n)$ we symmetry a $x^* = (x_1, \cdots, -x_n)$. Since $|x-y| = |x^*-y|$, $y \in \partial H$, we choose $\phi(x,y) = -K(x^*-y)$. Therefore, $G(x,y) = 0$ if y in ∂H, $-\Delta_y \phi(x,y) - \delta_{x^*} = 0$ in H, $G(x,y) = K(x,y) - K(x^* - y) = K(x-y) - K(x-y^*)$ ($|x^*-y| = |x-y^*|$) is symmetric.*

Example 2.15 (The Unit Ball). *This time we choose symmetric $x^* = \frac{1}{|x|^2}x$ so that $|x^*|\,|x| = 1$. Recall that $K(x,y) = C_n |x-y|^{2-n}$($n \geqslant 3$), by simple computation we put $G(x,y) = K(x,y) - |x|^{2-n} K(x^*-y)$; in 2 dimention $G(x,y) = K(x,y) - K(x^*,y) - K(x)$ to get the fine property that $G(x,y) = 0$ when y in $\partial B(0,1)$. It leaves to show that $G(x,y)$ is symmetric.*

2.7. **Energy Methods.**

Theorem 2.16 (Uniqueness for Dirichelet Problem). *If*

$$\begin{cases} -\Delta u &= 0 \quad in \quad \Omega \\ u &= 0 \quad in \quad \partial\Omega \end{cases},$$

Then $u = 0$.

Proof We have

$$\begin{aligned} 0 = \int_\Omega -\Delta u \cdot u dx &= \int_\Omega -\partial_j^2 u \cdot u dx \\ &= \int_\Omega \partial_j u \partial_j u dx - \int_{\partial\Omega} \frac{\partial u}{\partial \nu} u d\sigma \\ &= \int_\Omega |\nabla u|^2 \, dx. \end{aligned}$$

Therefore, $u \equiv C$ and C must be 0. □

Find the minimum value of $f : \mathbb{R}^n \to \mathbb{R}$ you can either look at points where $\nabla f = 0$ or consider the Hessian matirx $(\frac{\partial^2 f}{\partial_i \partial_j})_{i,j}$.

2.8. **Variation Methods.** Think of the minimizers of functional for the PDE.

Def 2.17. *The* ***action functional***

$$\mathcal{J}(u) = \int_\Omega \frac{1}{2} |\nabla u|^2 - f u dx.$$

Now we kook at $\min \mathcal{I}(u)$ for $u \in \mathcal{A}$, where $\mathcal{A} = \{u \in C^2(\Omega), u = g on \partial\Omega\}$. Obviously, $\mathcal{J}$ is convex (any liner function is convex) so it only has one critical point to reach the minimium, and suppose a minimum point u exists, we need to show $-\Delta u = f$.

Here we let $u_h = u + hv$, $h \in \mathbb{R}$, with $v \in C^2$, $v = 0$ on $\partial\Omega$. Now we consider $\mathcal{J}(u_h) = \mathcal{J}(u + hv)$, and $\frac{d}{dh}\mathcal{J}(u+hv)|_{h=0} = 0$.

$$\mathcal{J}(u_h) = \int_\Omega \frac{1}{2} |\nabla u|^2 + h \nabla u \nabla v + \frac{1}{2} |\nabla v|^2 u^2 - fu - hfv dx \tag{5}$$

$$\frac{d}{dh}\mathcal{J}(u+hv)|_{h=0} = \int_\Omega \nabla u \nabla v - f v dx = 0 \tag{6}$$

$$\int_\omega -\Delta u \cdot v - f v dx + \int_{\partial\Omega} \frac{\partial u}{\partial \nu} v d\sigma (= 0) = 0 \tag{7}$$

Therefore, we have $-\Delta u - f = 0$ $(\forall v)$, then $-\Delta u = f$. And we also have

$$\begin{aligned} D\mathcal{J}(u)\cdot v &= \int_\Omega (-\Delta u - f)v dx \\ D\mathcal{J}(u) &= -\Delta u - f. \end{aligned}$$

This means finding the cirtical point of $\mathcal{J}$ is finding the solution.

3. Solving Heat Equation

3.1. Foundamental Solution for the Heat Equation. The heat equation is

$$\begin{cases} (\partial_t - \Delta)u &= f \\ u(0,x) &= u_0(x) \end{cases}$$

Def 3.1. *Given $f = 0$ is a **homogeneous** heat equation, otherwise is a **inhomogeneous** equation, u is defined on $\mathbb{R}^+ \times \mathbb{R}^n$.*

Try to compute a foundamental solution $K(t,x)$ for $(\partial_t - \Delta)K = \delta_0$. K should satisfies some symmetries like

- Rigid spatial rotations: $K(t,x) = K(t,|x| = r)$.
- Scaling symmetries: $u(t,x)$ satisfies the euqation means $u(\lambda^2 t, \lambda x)$ still satisfies the euqation. If we have $K_\lambda(t,x) = \lambda^a K(\lambda^2 t, \lambda x)$,

$$\begin{aligned} (\partial_t - \Delta)K_\lambda(\Phi) &= K_\lambda((-\partial_t - \Delta)\Phi) \\ &= \int\int \lambda^a K(\lambda^2 t, \lambda x)(-\partial_t - \Delta)\Phi(t,x) dx dt \\ &= \int\int \frac{\lambda^a}{\lambda^{n+2}} K(s,y)[(-\partial_t - \Delta)\Phi](\frac{s}{\lambda^2}, \frac{y}{\lambda}) ds dy \\ &= \int\int \frac{\lambda^{a+2}}{\lambda^{n+2}} K(s,y)(-\partial_t - \Delta)\Phi_{\frac{1}{\lambda}}(s,y) ds dy \\ &= \lambda^{a-n}(\partial_t - \Delta)K\cdot\Phi_{\frac{1}{\lambda}} \\ &= \lambda^{a-n}\delta_0\Phi_{\frac{1}{\lambda}} = \lambda^{a-n}\Phi_{\frac{1}{\lambda}}(0,0). \end{aligned}$$

Therefore, $K_\lambda = \lambda^n K(\lambda^2 t, \lambda x)$ is also a foundamental solution $(\forall \lambda)$.

Now, setting $\lambda = \frac{1}{\sqrt{t}}$ we have $K(t,x) = t^{-\frac{n}{2}}K(1, \frac{x}{\sqrt{t}}) = t^{-\frac{n}{2}}\Psi(\frac{x^2}{t})$. Then

$$\begin{aligned} (\partial_t - \Delta)K &= \left(-\frac{n}{2}t^{-\frac{n}{2}-1}\Psi(\frac{r^2}{t}) - t^{-\frac{n}{2}}\Psi'(\frac{r^2}{t})\frac{x^2}{t^2}\right) + \left(t^{-\frac{n}{2}}(\partial_r^2 + \frac{n-1}{r}\partial_r)\Psi(\frac{r^2}{t})\right) \\ &= t^{-\frac{n}{2}-1}\left(-\frac{n}{2}\Psi(y) - \Psi'(y)y - 2\Psi'(y) - \Psi''(y)4y - \Psi'(y)2(n-1)\right) \\ &= t^{-\frac{n}{2}-1}\left(-\frac{n}{2}\Psi(y) - (y+2n)\Psi'(y) - 4y\Psi''(y)\right) = 0 \\ 0 &= \frac{n}{2}\Psi(y) + (y+2n)\Psi'(y) + 4y\Psi''(y) \\ &= y(4\Psi' + \Psi)' + \frac{n}{2}(4\Psi' + \Psi) \end{aligned}$$

Therefore we have $f = 4\Psi' + \Psi = Cy^{-\frac{n}{2}}$. f is not smooth unless $C = 0$ which means $\Psi(y) = Ce^{-\frac{y}{4}}$, then $K(t,x) = Ct^{-\frac{n}{2}}e^{-\frac{x^2}{4t}}$. Actually we can only consider $t \geqslant 0$ by the property of heat euqation to maike the function smooth.

Now we determine the constant for $K(t) = Ct^{-\frac{n}{2}}e^{-\frac{x^2}{4t}}$, and recall that if we have a smooth Φ with $\int \Phi dx = 1$, setting $\Phi_\epsilon = \epsilon^{-n}\Phi(\frac{x}{\epsilon})$, then $\Phi_\epsilon \to \delta_0$. Therefore,

$$\lim_{t\to 0} K(t) = C(\sqrt{t})^{-n}e^{-(\frac{x}{2\sqrt{t}})^2} = \delta_0$$

requires $\int Ce^{-(\frac{x}{2})^2}dx = 1$ which means $C = 2^n\pi^{\frac{n}{2}} = \frac{1}{(4\pi)^{\frac{n}{2}}}$. Therefore,

$$K(t,x) = \frac{1}{(4\pi t)^{\frac{n}{2}}}e^{-\frac{x^2}{4t}}1_{t\geqslant 0},$$

which is smooth except for $(0,0)$, $1_{t\geqslant 0}$ is called causality.

3.2. The Homogeneous Probelem.

Proposition 3.2. *A solution is given by $u(t) = K(t) * u_0$.*

Proof Since

$$(\partial_t - \Delta)u = \partial_t K * u_0 - \Delta K * u_0 = 0(t > 0),$$

and

$$\lim_{t\to 0} u(t) = \lim_{t\to 0} K(t) * u_0 = u_0,$$

therefore is proved.This also makes sense for $u_0 \in \mathcal{E}'$, and the solution is smooth for $t > 0$ (regularizing effect). □

3.3. Inhomogeneous Problem.

Proposition 3.3. *A solution is given by $u = K * f$.*

Proof At $t < 0$ and $t > 0$, $(\partial_t - \Delta)K = 0$. At $t = 0$, $(\partial_t - \Delta)K = \delta_0$, K jumps from 0 to $\delta_{x=0}$. Therefore $(\partial_t - \Delta)K = \delta_{t=0}\delta_{x=0} = \delta_{(t,x)=(0,0)}$ (non-rigorous notation). □

Proposition 3.4 (Duhamel Formula)**.**

$$u(t) = K(t) * u_0 + \int_0^t K(t-s) * f(s)ds.$$

Recall in ODE $x(t) = x_0e^{at} + \int_0^t e^{a(t-s)}f(s)ds$.

3.4. Mean Value Property for Heat Equation.

In $\mathbb{R}^{n+1}$,

$$\int (\partial_t - \Delta)u\Phi dxdt = \int (-\partial_t - \Delta)\Phi u dxdt.$$

We call $-\partial_t - \Delta$ an **adjoint heat operator** or **backward heat operator**.

$$K^{back}(t,x) = \frac{1}{(4\pi t)^{\frac{n}{2}}}e^{\frac{x^2}{4t}}1_{t\leqslant 0}$$

is supproted where $t \leqslant 0$. We will average over sets $E_r = \{(t,x) : K^{back}(t,x) \geqslant r^{-n}\}$ to get

$$u(0,0) = c\int_{E_r} u(t,x)w(t,x)dxdt.$$

Studying the **parabolic cylinder** where E_r lies in:

- has size located in $r^2 \times r$ (parabolic sealing, due to the operator of heat equation).
- backward in time.

Therefore, we have

$$\int_{E_r} (\partial_t - \Delta) u\Phi dxdt = \int_{E_r} (-\partial_t - \Delta)\Phi u dxdt + \int_{\partial E_r} \nu_t \cdot u\Phi - \nu_j \partial_j u\Phi + \nu_j u \partial_j \Phi d\nabla \tag{8}$$

To get rid of boundary, we want

1 $\Phi = 0$ on ∂E_r.
2 $\nabla\Phi = 0$ on ∂E_r.

We let $\Phi(t,x) = K^{back}(t,x) - r^{-n}$(satisfy 1)$-Cln(r^n K^{back}(t,x))$(satisfy 1 and 2). By putting $C = r^{-n}$ we get $\nabla\Phi = \nabla K^{back} - C\frac{r^n \nabla K^{back}}{1}$. Therefore we get

$$\int_{E_r} (-\partial_t - \Delta)\Phi u dxdt = 0 \tag{9}$$

$$= \int_{E_r} \left[\delta_{(0,0)} - r^{-n}\left(\frac{-\partial_t K^{back} - \Delta K^{back}}{K^{back}} + \frac{(\nabla K^{back})^2}{K^{back}}\right)\right] udxdt \tag{10}$$

$$= \int_{E_r} \left[\delta_{(0,0)} - r^{-n}\left(\nabla_x ln(K^{back})\right)\right] udxdt \tag{11}$$

$$u(0,0) = C\int_{E_r} w(t,x)(= \frac{|x|^2}{4t^2})u(t,x)dxdt. \tag{12}$$

A translation version is

$$u(t_0, x_0) = C\int_{E_r + (t_0,x_0)} \frac{|x-x_0|^2}{4(t-t_0)^2} u(t,x)dxdt.$$

3.5. **The Maximum Principle for the Heat Equation.** If we take the domain as $C_T = \Omega \times [0,T]$ which is a parabolic cylinder, then take **parabolic boundary** $\partial C_T = \Omega \times \{0\} \cup \partial\Omega \times [0,T]$ and leaves the head.

Theorem 3.5 (Strong Maximum Principle)**.** *If $u \in C^2(C_T) \cap C(\overline{C_T})$, then (Weak Maximum Principle)*

$$\max_{\overline{C_T}} u(t,x) = \max_{\partial C_T} u(t,x).$$

Furthermore, if (x_0, t_0) is a max point inside then u is constant up to time t_0.

Proof

1 Mean Value Property:

Suppose (t_0, x_0) is a max point inside, then we have

$$\max u = u(t_0,x_0) = C\int_{E_r+(t_0,x_0)} \frac{|x-x_0|^2}{4(t-t_0)^2} u(t,x)dxdt \leqslant \max u.$$

Therefore, for $(t,x) \in (t_0,x_0) + E_r$ we have $u(t,x) = u(t_0,x_0)$. Therefore, we can prove that the set where $u(t,x) = u(t_0,x_0)$ is open and close in $\Omega \times [0,t_0]$.

2 A more general proof but cannot be used to prove strong maximum principle: generally constant is not a trivial solution of PDEs.

Suppose (t_0,x_0) is a max point inside, then it's a critical point to get $\partial_t u(t_0,x_0) = 0$, $\partial_x u(t_0,x_0) = 0$. It's a maximum point so $\Delta u(t_0,x_0) \leqslant 0$ (think about $-x^2$). Therefore, $(\partial_t - \Delta)u(t_0,x_0) \geqslant 0$.

If (t_0, x_0) is on the top we have $\partial_t u(t_0, x_0) \geqslant 0$ to get $(\partial_t - \Delta)u(t_0, x_0) \geqslant 0$.

Then we want to get contradiction. If we fix $u_\epsilon = u - \epsilon t$ we have $(\partial_t - \Delta)u_\epsilon = -\epsilon$ which means it cannot have a maximum point inside. Therefore, it's maximum can only be reached on ∂C_T:

$$\max_{\overline{C_T}} u_\epsilon = \max_{\partial C_T} u_\epsilon.$$

As $\epsilon \to 0$, $u_\epsilon \to u$ uniformly in $\overline{C_T}$.

Note: for strict subsolution of heat equation, we have strong maximum principle.

□

3.6. **Initial or Boundary Value Problems for the Heat Equation.**

Given

$$\begin{cases} (\partial_t - \Delta)u &= f \\ u(0,x) &= u_0(x) \\ u(t,x) &= g(t,x) \quad (t,x) \in \partial\Omega \times [0,T] \end{cases}$$

Alternatively, we give the rate in which heat is transmitted as

$$\frac{\partial u}{\partial \nu}(t,x) = g(t,x), (t,x) \in \partial\Omega \times [0,T],$$

which is called as Neumann boundary condition.

Theorem 3.6. *Min/Max principles give uniqueness for the Dirichlet boundary problem. Note: only gives uniqueness. Existence proof occurs in 1980s by finding the middle of super and sub solutions.*

Theorem 3.7. *Solutions to the heat equation are C^∞.*

Proof Change notations in the proof for Δ using the fact that fundamental solution is smooth away from $(0,0)$. □

3.7. **Energy Estimates for the Heat Equation.**

This is used to get proof of existence. e define 'energy' of the solution as

$$\int_\Omega |u(t,x)|^2 \, dx.$$

Then the **energy dissipation** is

$$\begin{aligned} \frac{d}{dt}\int_\Omega |u(t,x)|^2 \, dx = 2\int_\Omega u\partial_t u dx = 2\int_\Omega u\Delta u dx \\ = -2\int_\Omega |\nabla u|^2 \, dx + 2\int_{\partial\Omega} u\frac{\partial u}{\partial \nu} dx \leqslant 0, \end{aligned}$$

for $u = 0$ resp. $\frac{\partial u}{\partial \nu} = 0$ for Dirichilet problem resp. Neumann problem. If $u(0) = 0$ then because of $\frac{d}{dt}\int_\Omega |u(t,x)|^2 \, dx \leqslant 0$, $\int_\Omega |u(t,x)|^2 \, dx \equiv 0$ which means $u \equiv 0$.Many other energies:

$$E(u) = \int |\Delta u|^2 \, dx, E(u) = \int |u|^p \, dx, 1 < p < +\infty.$$

Actual thermal energy:

$$E(u) = \int_\Omega u dx$$
$$\frac{d}{dt}E(u) = \int_\Omega 1\Delta u dx = \int_{\partial\omega} \nabla u \cdot \nu dx = \int_{\partial\omega} \frac{\partial u}{\partial \nu} dx.$$

The energy is conserved in Neumann problem when $\frac{\partial u}{\partial \nu} = 0$. Therefore, Heat euqation has properties like

- energy dissipation
- infinite speed of propagation (Every value on the boudary will get an smooth function)

4. Solving Wave Equation

The wave equation is

$$\begin{cases} (\partial_t^2 - \Delta)u &= f \\ u(0,x) &= u_0(x) \\ u_t(0,x) &= u_1(x) \end{cases}$$

There are a lot of models like

- vibrating string (1-d)
- drum (2-d)
- elastic waves
- sound waves
- electromagnetic waves
- gravitational waves
- NOT water waves
- compressible fluids

4.1. **Wave equation in 1-d.**

In 1-d,

$$(\partial_t^2 - \partial_x^2) = (\partial_t - \partial_x)(\partial_t + \partial_x).$$

Therefore we study transportation euqations

$$(\partial_t - \partial_x)v = 0, v = (\partial_t + \partial_x)u, \tag{13}$$

$$(\partial_t + \partial_x)w = 0, w = (\partial_t - \partial_x)u. \tag{14}$$

Since v gets transported in direction $(-1,1)$, we have

$$v(x,t) = v(x+t,0) = u_1(x+t,0) + \partial_x u_0(x+t,0),$$

$$w(x,t) = w(x-t,0) = u_1(x-t,0) - \partial_x u_0(x-t,0).$$

Remark: Any solution to the homogeneous wave euqations can be represented as

$$u(x,t) = F(x+t)(\text{with speed 1 to left}) + G(x-t)(\text{with speed 1 to right}),$$

both of them are plane waves.Since

$$\partial_t u = \frac{1}{2}(v+w)\partial_x u = \frac{1}{2}(v-w),$$

we have $F' = \frac{1}{2}v$, $G' = -\frac{1}{2}w$. Then we have

$$\begin{aligned} \partial_x u(x,t) &= \frac{1}{2}(v-w) = \frac{1}{2}(v(x+t,0) - w(x-t,0)) \\ u(x,t) &= \int_{-\infty}^{x} \partial_x u(x,t) = \frac{1}{2}(u_0(x+t) + u_0(x-t)) + \frac{1}{2}\int_{-\infty}^{x} u_1(y+t) - u_1(y-t)dy \\ &= \frac{1}{2}(u_0(x+t) + u_0(x-t)) + \frac{1}{2}\int_{x-t}^{x+t} u_1(z)dz, \end{aligned}$$

which means for u_0 we only use the end points and for u_1 we use the entire interval.Key properties:

- Finite speed of propagation: waves only wave with speed 1 in all directions.

4.2. **Find a FORWARD fundamental solution.** How do we choose the initial date to get a fundamental solution? Take

$$\tilde{u} = \begin{cases} u & t > 0 \\ 0 & t < 0 \end{cases}$$

Therefore,

$$\Delta\tilde{u} = \begin{cases} \Delta u & t > 0 \\ 0 & t < 0 \end{cases},$$

$$\partial_t\tilde{u} = \begin{cases} \partial_t u & t > 0 \\ 0 & t < 0 \end{cases} + \delta_{t=0}u_0(x),$$

$$\partial_t^2\tilde{u} = \begin{cases} \partial_t^2 u & t > 0 \\ 0 & t < 0 \end{cases} + \delta_{t=0}u_1 + \delta'_{t=0}u_0.$$

Therefore if $u_0 = 0$, $u_1 = \delta_{x=0}$ then we get $(\partial_t^2 - \Delta)\tilde{u} = \delta_{t=0,x=0}$. Then in 1-d

$$K(x,t) = u(t,x) = \frac{1}{2}1_{[-t,t]}(x).$$

The Midterm 2

4.3. **Wave equations in higher dimensions: symmetries for $\Box$.** We define $\Box = \partial_t^2 - \Delta_X$, then we have the wave equation as

$$\begin{cases} \Box u & = & f \\ u(0,x) & = & u_0(x) \\ u_t(0,x) & = & u_1(x) \end{cases}$$

- translation symmetry (constant coeffient)
- rigid spatial rotation.

  If we put $t = x_0$, we may represent $\Box u = -M_{\alpha,\beta}\partial_\alpha\partial_\beta u$, with

$$M = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}.$$

  Thereforem if we put $y = Ax$, then

$$\Box y = A^T MAx \Rightarrow A^T MA = M.$$

  The set of all A forms a group L named Lorentz group. We want

$$A = \begin{pmatrix} 1 & \\ & Q \end{pmatrix} \text{(rigid spatial rotation)}.$$

  with Q is a spatial rotation which means it's orthogonal.

Example 4.1. *In $1+1$ dimension, we have*

$$A_1 = \begin{pmatrix} cos(\theta) & sin(\theta) \\ -sin(\theta) & cos(\theta) \end{pmatrix},$$

Lorentz boosts as

$$A_2 = \begin{pmatrix} cosh(\theta) & sinh(\theta) \\ sinh(\theta) & cos(\theta) \end{pmatrix}.$$

which keeps the quadiratic form $X^T M X = x_0^2 - x_1^2$ unchanged.

Theorem 4.2. *Any elemet of Lorentz group L is a conposition of the rigid spatial rotation and a Lorentz boost.*

Now we look for fundamental solution that

- forward in time
- lorentz invariant which means it keeps $t^2 - x^2 = \text{Const}$ so $K = K(t^2 - x^2)$.

In $1-d$ we have $K = \frac{1}{2}H(t^2 - x^2)H(t)$. In higher dimensions, we have the fundamental solution supported in where $t > 0$, $t^2 - x^2 > 0$. By homogeneity, we gave K be homogeneous of order $1-n$ (compared to Δ), then $K = (t^2 - x^2)^{\frac{1-n}{2}}$, and it's integrable only for $n < 3$ which means it can be thought as a distribution.

4.4. Fundamental solutions in 2-d.

Theorem 4.3. *If $n = 2$, then $K = C_2 \frac{1}{\sqrt{t^2 - x^2}} 1_{t \geqslant 0}$.*

In order to find C_2, we have the wave equation as

$$\begin{cases} \Box K &= 0 \\ K(0,x) &= 0 \\ K_t(0,x) &= \delta_0(x) \end{cases}$$

Therefore, by taking derivatives of $\int K dx$, we have

$$\begin{aligned} \frac{d}{dt} \int K dx &= \int K_t dx = 1(t = 0) \\ \frac{d^2}{dt^2} \int K dx &= \int \partial_t^2 K dx = \int 1 \Delta K dx = 0 \\ \int K dx &= \int K(0,x) dx + t \int K_t(0,x) dx = 0 + t = t \\ t &= C_2 \int_{|x| \leqslant t} \frac{1}{\sqrt{t^2 - x^2}}. \end{aligned}$$

If we set $t = 1$ we have $C_2 = \frac{1}{2\pi}$.

4.5. Fundamental solutions in higer dimensions.

For Higher dimensions where $n \geqslant 3$, we have $K(t,x) = \phi(t^2 - x^2)$ and supported in $t^2 - x^2 > 0$.

$$\Box \phi(t^2 - r^2) = 4(t^2 - r^2)\phi'' + 2(n+1)\phi' = 0,$$

$$\Box \phi(y) = 2y\phi'' + 2(n+1)\phi' = 0. \tag{15}$$

This means

$$\begin{aligned} \phi &= C(t^2 - x^2)^{\frac{1-n}{2}}, \\ \phi(y) &\approx \frac{1}{y^{\frac{n-1}{2}}}. \end{aligned}$$

Def 4.4. *We define the **principal value** of an integral like $\phi(\psi)$:*

$$p.v\phi(\psi) = \lim_{\epsilon \to 0} \frac{\psi(y)}{y} dy.$$

In this way, ϕ can be viewed as a distribution and equals δ_0 (δ_0 solves euqation (15)).

When $n = 3$, we have $K(t, x) = c\delta_{t^2-x^2=0}$, supported on where $t = |x|$. This means

$$\delta_{t^2-x^2=0} = \frac{1}{t+|x|}\delta_{t-|x|=0} = \frac{1}{2t}\delta_{t-|x|=0},$$

which means $K(t, x) = \frac{c}{t}\delta_{t=|x|}$.